Conformal Field Theory and the Long-Range Ising Model

ICFP M1 Library-Based Project

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Context

- Career aspirations in theoretical high energy physics
- M1 internship at the University of Turin with Lorenzo Bianchi, in the *String theory and Supergravity* group
- Conformal field theory and defects

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The Big Picture

The long-range Ising model: a rich theory which will take us on a journey through CFT and QFT!

$$H = -J \sum_{i \neq j} \frac{s_i s_j}{|i-j|^{d+\sigma}} \tag{1}$$

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$$H = -J \sum_{i \neq j} \frac{s_i s_j}{|i-j|^{d+\sigma}} \tag{1}$$

- Study the system at its second-order phase transition
- Calculate correlators in the critical theory
- Show that the critical theory is a CFT following Paulos et al. [1]

The Big Picture

Renormalization group philosophy

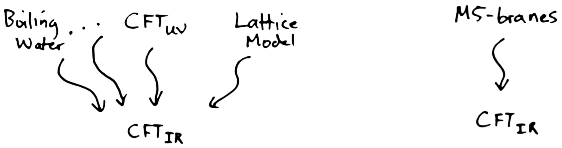

Figure: Sketch of distinct microscopic UV theories flowing to the same IR CFT from [2].

- Study QFT by constraining RG flows between UV and IR
- If conformal symmetry at fixed points, tools from CFT drastically simplify correlation functions
- Infinite correlation length at critical points implies scale invariance → hint of CFT?

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Conformal transformations

Definition

A transformation $x \to \tilde{x}$ is said to be conformal if the metric transforms in the following way:

$$g_{\mu\nu}(x) \to \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} g_{\alpha\beta}(x) = \Lambda(x) g_{\mu\nu}(x) \tag{2}$$

The conformal group in $d \geq 3$

- $\begin{cases} \text{Translations} \\ \text{Rotations} \end{cases} \rightarrow \mathbb{R}^{p,q} \rtimes O(p,q)$

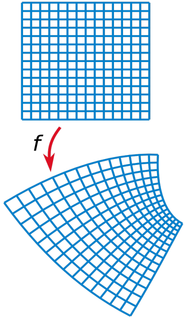

Figure: Sketch of a conformal mapping from the Wikipedia page on conformal transformations.

The conformal group in $d \geq 3$

- $\left.\begin{cases}\text{Translations}\\ \text{Rotations}\end{cases}\right. \rightarrow \mathbb{R}^{p,q} \rtimes O(p,q)$
- Dilatations
- Special conformal transformations

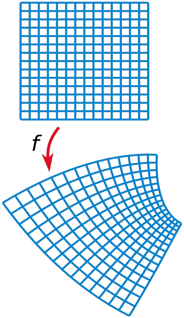

Figure: Sketch of a conformal mapping from the Wikipedia page on conformal transformations.

The conformal algebra - commutation relations

$$\begin{cases} [D, P_\mu] = iP_\mu \\ [D, K_\mu] = -iK_\mu \\ [K_\mu, P_\nu] = 2i(\eta_{\mu\nu} D - L_{\mu\nu}) \\ [K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu) \\ [P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu) \\ [L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho}) \end{cases} \tag{3}$$

Note that P and K act like ladder operators on eigenstates of D!

These relations can be cast into the following compact form:

$$[J_{mn}, J_{pq}] = i\left(\eta_{mq}J_{np} + \eta_{np}J_{mq} - \eta_{mp}J_{nq} - \eta_{nq}J_{mp}\right) \tag{4}$$

Conformal algebra

The Lie algebra of the conformal group is given by $\mathfrak{so}(p+1, q+1)$ in a metric η_{mn} of signature (p, q).

Studying representations yields to following transformation rule:

Transformation law for quasi-primaries

A primary field transforms in the following way under finite conformal transformations:

$$\phi(x) \to \phi'(x') = \left|\frac{\partial \tilde{x}}{\partial x}\right|^{-\Delta/d} \phi(x) = |\Lambda(x)|^{\Delta/d} \phi(x) \tag{5}$$

Constraints on correlation functions for primary operators

Two and three-point functions

Given three local primaries $\mathcal{O}_1(x_1)$, $\mathcal{O}_2(x_2)$, $\mathcal{O}_3(x_3)$ of respective scaling dimensions Δ_1, Δ_2, Δ_3 :

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{\delta_{ij}}{|x_1 - x_2|^{2\Delta_{i,j}}} \tag{6}$$

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{\lambda_{123}}{|x_1 - x_2|^{h_{123}}\,|x_1 - x_3|^{h_{131}}\,|x_2 - x_3|^{h_{231}}} \tag{7}$$

where $h_{ijk} = \Delta_i + \Delta_j - \Delta_k$.

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where $h_{ijk} = \Delta_i + \Delta_j - \Delta_k$.

Remark: The collection of Δ_i's and λ_{ijk}'s, or CFT data, completely characterize all correlators of a CFT!

Operator product expansion (OPE) Product of two local operators expressed as a linear combination of all of the operators of a given theory.

$$\mathcal{O}_i(x)\mathcal{O}_j(y) = \sum_k f_{ijk}(z)\mathcal{O}_k(z) \tag{8}$$

OPE in CFT

Again, CFT imposes the form OPEs can take:

$$\mathcal{O}_i(x)\mathcal{O}_j(y) = \sum_k \frac{\lambda_{ijk}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}}\mathcal{O}_k(x-y) \tag{9}$$

where λ_{ijk} are the three-point function data.

Stress tensor criterion for conformal invariance:

$$T^{\mu}{}_{\mu} = 0 \tag{10}$$

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LRI lacks a stress tensor! Weaker definition of CFT: theory satisfying global Ward identities

Global Ward identities: Using correlators, with G_a generator of a conformal symmetry

$$\sum_{i=1}^{n} \langle \phi(x_1) \dots G_a \phi(x_i) \dots \phi(x_n) \rangle = 0 \tag{11}$$

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Quantum Field Theory Essentials

Path integral formulation: Formulation of QFT complementary to that of canonical quantization.

$$\langle \phi_1(x_1)\dots\phi_n(x_n)\rangle = \frac{1}{Z_0}\int \mathcal{D}\Phi\,\phi_1(x_1)\dots\phi_n(x_n)e^{iS[\Phi]} \tag{12}$$

Euclidean path integral

Using a Wick rotation (imaginary time) and hence working in Euclidean space:

$$\langle \phi_1(x_1)\dots\phi_n(x_n)\rangle = \frac{1}{Z_0}\int \mathcal{D}\Phi\,\phi_1(x_1)\dots\phi_n(x_n)e^{-S[\Phi]} \tag{13}$$

Quantum Field Theory Essentials

Typical action in this project:

$$S[\phi] = \frac{1}{2} \int d^d x d^d y \phi(x) D(x, y) \phi(y) + \frac{\lambda}{4!} \int d^d x \phi(x)^4 \tag{14}$$

But first...

$$S_0[\phi] = \frac{1}{2} \int d^d x d^d y \phi(x) D(x, y) \phi(y) \tag{15}$$

This is a Gaussian theory, ϕ is referred to as a *generalized free field* (GFF).

Correlation functions in the free theory

Wick's theorem

n-point correlation functions in a Gaussian theory are given by:

$$\langle \phi(x_1) \ldots \phi(x_n) \rangle_0 = \sum_{\text{pairings}} G(x_{i_1} - x_{i_1}) \ldots G(x_{i_{n-1}} - x_{i_n}) \tag{16}$$

where $G(x - y) := D^{-1}(x, y)$ is the Green's function of the D operator.

No such result for interacting theories unfortunately...

Quantum Field Theory Essentials

No such result for interacting theories unfortunately...
Standard approach:

Perturbation theory

Treating coupling as "small", and abusively permutating integral and summation signs:

$$\langle\phi(x_1)\dots\phi(x_n)\rangle = \frac{\langle\phi(x_1)\dots\phi(x_n)e^{-\frac{\lambda}{4!}\int d^dx\phi^4}\rangle_0}{\langle e^{-\frac{\lambda}{4!}\int d^dx\phi^4}\rangle_0}$$
$$= \frac{\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^k}{4!^k k!}\int d^dy_1\dots d^dy_k\langle\phi(x_1)\dots\phi(x_n)\phi^4(y_1)\dots\phi^4(y_k)\rangle_0}{\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^k}{4!^k k!}\int d^dy_1\dots d^dy_k\langle\phi^4(y_1)\dots\phi^4(y_k)\rangle_0} \tag{17}$$

Feynman diagrams

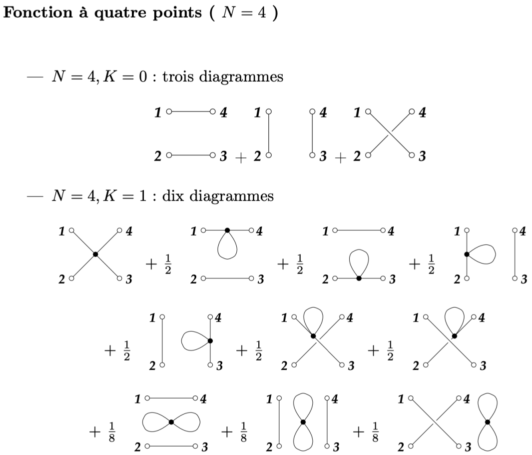

Figure: Taken from [3].

Quantum Field Theory Essentials

A priori divergent diagrams...

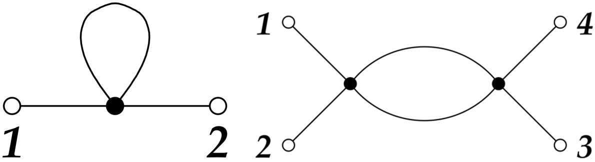

Figure: Left: divergent "tadpole" diagram in first order expansion of two-point function. Right: divergent diagram in second order expansion of four-point function [3].

One-loop renormalization: Previous diagrams typically have poles in a quantity ε related to dimension d.

One-loop renormalization: Previous diagrams typically have poles in a quantity ε related to dimension d.
Try to get rid of them using counter-terms:

$$S = \frac{1}{2}\int d^d x d^d y \left(Z_1^{\frac{1}{2}}[\phi]\right) D(x,y) \left(Z_1^{\frac{1}{2}}[\phi]\right) + \int d^d x \frac{Z_\lambda \lambda}{4!} Z_4[\phi^4] \quad (18)$$

where $[\phi]$, $[\phi^4]$ are the renormalized fields and the Z-factors are expressed as a Laurent series in ε:

$$Z_A = 1 + \sum_{k=1}^{+\infty} \frac{f_k^A}{\varepsilon^k} \quad (19)$$

Quantum Field Theory Essentials

Other point of view: **the renormalization group**.

- Equip theory with an energy scale (UV cutoff) μ
- Starting with an action at scale μ_0, one can derive an effective action at scale μ_1
- Renormalization factors account for effective interactions at a given scale
- The bare theory's fields are written as a function of the renormalized theory's fields: e.g. $\phi^n = Z_n[\phi^n]$.

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Callan-Symanzik equation

Defining the dimensionless coupling g using a power of μ and expressing scale invariance of the bare theory leads to the following Callan-Symanzik equation:

$$\left[\frac{\partial}{\partial \ln \mu} + \beta_i \frac{\partial}{\partial g} + n\gamma\right] G_\mu(x_1, \ldots, x_n) = 0 \tag{20}$$

where $\gamma = \frac{1}{Z}\frac{\partial Z}{\partial \ln \mu}$ (anomalous dimension) and $\beta(g) = \frac{\partial g}{\partial \ln \mu}$.

In this work, β-functions go like

$$\beta(g) = -\varepsilon g + K g^2 + \mathcal{O}(g^3) \tag{21}$$

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Two fixed points i.e. zeros of β:

- $g = 0$ (Gaussian UV fixed point)
- $g = \varepsilon/K$ (Wilson-Fisher IR fixed point)

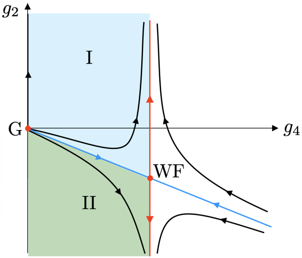

Figure: Renormalization group flow diagram of massive ϕ^4 theory [4].

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The Long-Range Ising Model

Take continuum limit (lattice spacing $a \to 0$):

$$H = -J(a) \int d^d x d^d y \frac{\phi(x)\phi(y)}{|x-y|^{d+\sigma}} \tag{22}$$

Continuum theory

Switch to action picture, perturb the corresponding action by ϕ^4 interaction:

$$S = S_0 + S_1 = \frac{\mathcal{N}_\sigma}{2} \int d^d x \phi(x) \mathcal{L}_\sigma \phi(x) + \frac{g_0}{4!} \int d^d x \phi(x)^4 \tag{23}$$

where $\mathcal{L}_\sigma = (-\partial^2)^{\sigma/2}$ is the fractional Laplacian.

Remark: This is a GFF perturbed by a ϕ^4 interaction!

The Long-Range Ising Model

Three cases for the critical theory:

- $\sigma < d/2$: the critical theory is Gaussian
- $d/2 < \sigma < \sigma^* = 2 - \eta_{\text{SRI}}$: the critical theory is non-trivial, non-Gaussian
- $\sigma^* < \sigma$: the critical theory is the short-range Ising model (SRI)

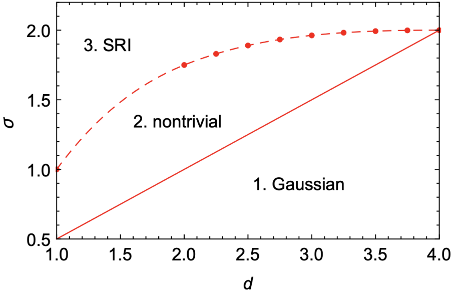

Figure: Critical theory as function of d, σ [1].

Conformal invariance in the LRI

The critical theory is conformally invariant. In this work, we focus on proving conformal invariance near the $\sigma = d/2$ crossover, the so-called LRI fixed point.

The Gaussian theory: One can show that the Gaussian (unperturbed) action is conformally invariant.

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Otherwise, look at propagator (modulo rescaling by $\mathcal{N}_\sigma$):

$$G(x) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip\cdot x}}{|p|^\sigma} = \frac{2^{d-\sigma}\Gamma\left(\frac{d-\sigma}{2}\right)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{\sigma}{2}\right)} \frac{1}{|x|^{d-\sigma}} \tag{24}$$

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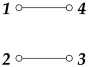
This is consistent with the constraints of CFT and yields $\Delta_\phi = (d-\sigma)/2$.
As a sanity check, one can study a three-point function:

$\langle \phi(x_1)\phi(x_2)\phi^2(x_3)\rangle =$



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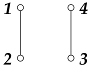

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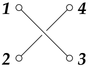

$$\begin{aligned} &= G(x_1 - x_3)G(x_2 - x_3) + G(x_1 - x_3)G(x_2 - x_3) \\ &= \frac{2}{|x_1 - x_3|^{2\Delta_\phi}|x_2 - x_3|^{2\Delta_\phi}} \end{aligned} \tag{25}$$

The Long-Range Ising Model

ε-**expansion**: We look at the theory for $\sigma = (d + \varepsilon)/2$ for $\varepsilon \ll 1$ and define a dimensionless coupling g such that $g_0 = g\mu^{\varepsilon}$.

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One-loop divergences occur in the following expansion:

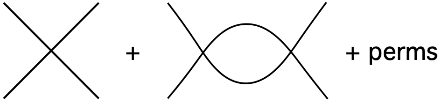

Figure: Expansion of the four-point function of ϕ to $\mathcal{O}(g^2)$.

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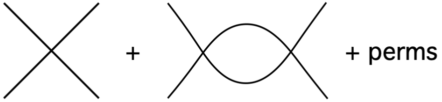

Figure: Expansion of the four-point function of ϕ to $\mathcal{O}(g^2)$.

Divergent contribution to the four-point function of ϕ:

$$72 \times \frac{1}{2} \times \frac{g^2 \mu^{2\varepsilon}}{4!^2} \int_{|x| \ll 1} \frac{d^d x}{|x|^{d-\varepsilon}} = 36 \frac{g^2}{4!^2} \frac{S_d}{\varepsilon} + \text{finite term} \tag{26}$$

β-function

The GFF + ϕ^4 theory's β-function up to $\mathcal{O}(g^3)$ is given by:

$$\beta(g) = -\varepsilon g + K g^2 = -\varepsilon g + \frac{3 S_d g^2}{2} \tag{27}$$

The Long-Range Ising Model

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NB: ϕ^n also requires renormalization for $n > 1$! Divergences occur in $\langle \phi^n \phi^n \rangle$. Leads to $\phi^n = Z_n[\phi^n]$ and $\gamma_n(g)$.

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Tests of conformal invariance

Recall for two primaries $\mathcal{O}_1$, $\mathcal{O}_2$:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{\delta_{ij}}{|x_1 - x_2|^{2\Delta_{i,j}}} \tag{28}$$

Correlators of operators whose Δ's do not differ by even integer should vanish.

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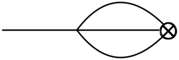
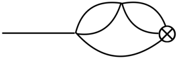
CFT test: $\langle \phi\phi^3 \rangle$, $\langle \phi^2\phi^4 \rangle$ should vanish

Tests of conformal invariance

Non-renormalized:

$$F_0(x) := \langle \phi(x)\phi^3(0)\rangle =$$  $$+$$ 

$$= \frac{R_1 g_0}{|x|^{d-2\varepsilon}} + \frac{R_2 g_0^2}{|x|^{d-3\varepsilon}} + \mathcal{O}(g_0^3) \tag{29}$$

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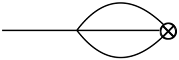

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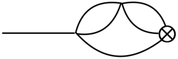

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Renormalized ($s = \mu|x|$):

$$F(x) = \frac{gR_1 s^\varepsilon + g^2 \left[R_1(K + K_3)\varepsilon^{-1} s^\varepsilon + R_2 s^{2\varepsilon} \right]}{|x|^{d-\varepsilon}} \tag{30}$$

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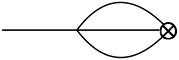

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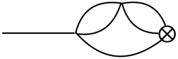

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Plugging in values of constants at IR, to second order in g:

$$F(x) = 0 \tag{31}$$

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Perturbative conformal invariance

Defect QFT

$$S = \frac{1}{2}\int d^{\bar{d}}X\,(\partial_M\Phi)^2 + \frac{g_0}{4!}\int_{y=0} d^d x\Phi^4 \tag{32}$$

where $\bar{d} = d + 2 - \sigma$ and $\Phi(x,0) = \phi(x)$.

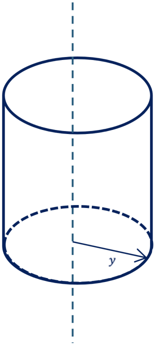

Figure: Bulk theory lives in $y > 0$ region, defect on $y = 0$ axis.

Perturbative conformal invariance

The bulk theory has a stress tensor and conformal currents!

$$T_{MN} = \partial_M \Phi \partial_N \Phi - \frac{1}{2}\delta_{MN} \left(\partial_K \Phi\right)^2 - \delta^{||}_{MN} \delta^{(p)}(y) \frac{g_0}{4!} \Phi^4 \tag{33}$$

Perturbative conformal invariance

The bulk theory has a stress tensor and conformal currents!

$$T_{MN} = \partial_M \Phi \partial_N \Phi - \frac{1}{2}\delta_{MN}\left(\partial_K \Phi\right)^2 - \delta^{||}_{MN}\delta^{(p)}(y)\frac{g_0}{4!}\Phi^4 \tag{33}$$

Inserting their divergences into n-point functions $\rightarrow$ broken Ward identities:

$$\sum_{i=1}^{n}\left[X_i \cdot \partial_{X_i} + \Delta_\phi\right] G(X_1, \ldots, X_n) = \beta(g)\frac{\mu^\varepsilon}{4!}\int d^d x G(X_1 \ldots X_n; \left[\phi^4\right](x)) \tag{34}$$

$$\begin{aligned}\sum_{i=1}^{n}\left[\left(2X_i^\mu X_i^\lambda - \delta^{\mu\lambda}X_i^2\right)\frac{\partial}{\partial X_i^\mu} + 2\Delta_\phi X_i^\lambda\right] G(X_1, \ldots, X_n) \\ = 2\beta(g)\int d^d x x^\lambda G(X_1 \ldots X_n; \left[\phi^4\right](x))\end{aligned} \tag{35}$$

Perturbative conformal invariance

Assuming that $g \to g_*$ and $y \to 0$ can be swapped (non trivial, further justified in [1]), conformal Ward identities hold at $g = g_*$ and hence the LRI fixed point is conformally invariant!

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Conclusion

- LRI motivated a brief overview of CFT, QFT including renormalization group
- Two interesting crossover points: the LRI and SRI fixed points. Extensively reviewed the proof of conformal invariance in the LRI fixed point in [1].
- Complementary picture presented in [5, 6] not presented in detail
- Further avenues to explore: CFT in $d = 2$ dimensions, the AdS/CFT correspondence, defects and the conformal bootstrap (see [7] for bootstrap and LRI)

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